1. Introduction

In physics and mathematics, Green’s theorem gives the relationship between a line integral around a simple closed curve $C$ and a double integral over the plane region $D$ bounded by $C$. This theorem is an application of the fundamental theorem of calculus for integrating a certain combination of derivatives over a plane. As both sides of its equality are finitely additive and almost all planar regions can be divided into triangles and rectangles, so that the result holds for any planar region practically all of which can be divided into triangles and rectangles. This proves the theorem for reasonably shaped regions. Its generalization to the non-planar surfaces is proved directly from it by using the finite additivity of both sides.

1.1 Green’s Theorem

The formal statement of Green’s theorem is as follows: Let $S$ be a sufficiently nice region in the plane, and let $\partial S$ be its boundary. Then, we have

$$\oint_{\partial S} (\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}) \, dx \, dy = \iint_{S} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) \, dx \, dy$$

where the boundary, $\partial S$, is traversed counterclockwise on its outside cycle (and clockwise on any internal cycles as you can be verified using zippers).

Meaning of this Theorem interpretation: Green’s Green’s theorem is a form that the fundamental theorem of calculus takes in the context of integrals over planar regions.

For a rectangle: By Using the ordinary fundamental theorem of calculus, we have;
For a right triangle: if, for convenience, we choose a triangle bounded by line $x = 0$, $y = 0$, and a right triangle bounded by line $x = a$, $y = b$. We similarly obtain:

$$
\int_{y=0}^{y=b} \int_{x=0}^{x=a} \frac{\partial f(x, y)}{\partial x} \, dx \, dy - \int_{y=0}^{y=b} \int_{x=0}^{x=a} \frac{\partial f(x, y)}{\partial y} \, dy \, dx = \int_{x=0}^{x=a} \left[ (y_2(b, y) - y_1(a, y)) \, dy - \int_{y=0}^{y=b} (\eta_2(x, y) - \eta_1(x, y)) \, dy \right]
$$

Rearrangement of right hand side gives the

Th**eorem for rectangles and right triangles** is obtained by rearranging the right hand side of the equation.

It means that Thus, for $R$, a rectangle or right triangle in the $x$-$y$ plane, (for which $dS = dS_k$), we have

$$
\int_{R} \nabla \times \mathbf{V} \, dS = \oint_{C} \mathbf{V} \times d\mathbf{l}
$$

Both sides of this equation is finite, are finitely additive, that is, if we evaluate either side over we take two disjoint regions, and evaluate either one over both, you get the result will be equal to the sum of their values the result of separate evaluations on the two regions separate. This is true even if the regions share a common boundary because the line integrals will cancel out over the common boundary which that ceases to be a boundary.

The result follows from additivity for any region that can be broken up divided into rectangles and triangles, which accounts for most regions we will encounter.