1. Introduction

In physics and mathematics, Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the region D bounded by C. This theorem is an application of the fundamental theorem of calculus to integrating a certain combination of derivatives over a plane. It can be proven easily for rectangular and triangular regions. As both sides of its equality are finitely additive, almost all planar regions can be divided into triangles and rectangles, so that the result holds for any planar region practically all of which can be divided into triangles and rectangles. This proves the theorem for reasonably shaped regions. It's its generalization to the non-planar surfaces is the Stokes' theorem described below.

1.1 Green's Theorem

The formal statement of Green's theorem is as follows. Let S be a sufficiently nice region in the plane, and let ∂S be its boundary. Then, we have:

\[ \oint_{\partial S} (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \, dx \, dy = \iint_{S} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \, dx \, dy \]

where the boundary, ∂S, is traversed counterclockwise on its outside cycle, and clockwise on any internal cycles as you can verify using zippers.

Meaning of this Theorem interpretation: Green's theorem is a form that the fundamental theorem of calculus takes in the context of integrals over planar regions.

For a rectangle: By the ordinary fundamental theorem of calculus, we have:
For a right triangle: If for convenience, we choose a triangle bounded by line \( x = 0, y = 0 \) and \( \frac{x}{a} + \frac{y}{b} = 1 \).

We similarly get:

\[
\begin{align*}
\int_{x=0}^{x=a} \int_{y=0}^{y=b} \left( \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} \right) \, dx \, dy &= \int_{x=0}^{x=a} \int_{y=0}^{y=b} \frac{\partial H}{\partial y} \, dy \, dx \\
&= \int_{y=0}^{y=b} \left( v_2(a-y/b, b, y) - v_1(0, y) \right) \, dy \, dx \\
&= \int_{x=0}^{x=a} \left( v_1(x, b - bx/a) - v_1(x, 0) \right) \, dx
\end{align*}
\]

Rearrangement of the right hand side gives the Theorem for rectangles and right triangles.

It means that, for \( R \), a rectangle or right triangle in the \( x-y \) plane, (for which \( dS = dS_k \)), we have

\[
\int_{R} \nabla \times \mathbf{v} \cdot dS = \oint_{\partial R} \mathbf{v} \cdot d\mathbf{l}
\]

Both sides of this equation are finite, additive, i.e., if we take two disjoint regions, and evaluate either one over both, you get the sum of their values on the two regions separate—separately.

This is true even if the regions share a common boundary, because the line integrals will cancel out over the common boundary which that ceases to be a boundary.

The result follows from additivity for any region that can be broken up into rectangles and triangles, which accounts for most regions we will encounter.